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ALGORITHM OF AUTOMATIC DETECTION AND ANALYSIS OF NON-EVOLUTIONARY CHANGES IN ORBITAL MOTION OF GEOCENTRIC OBJECTS

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The goal of this work is developing the methods and algorithms for automatic detection and analysis of non-evolutionary changes in orbital motion of geocentric objects caused by maneuvers and other events (but not by natural perturbations). The task is formulated for three kinds of non-evolutionary changes, namely the single impulse maneuver, two impulses maneuver and small continuous thrust. The methods and algorithms are developed. Examples of the algorithm work are given.

INTRODUCTION

Automatic determination and adequate characterization of the maneuvers performed by spacecraft in near-Earth orbits is a significant component of the general problem of tracking Earth satellites and catalog maintenance. This paper presents the techniques and algorithms for the detection and analysis of non-evolutionary changes of orbital motion of satellites resulting from maneuvers. The problem is formulated for three types of non-evolutionary change of orbital parameters: one burn maneuvers, two burn maneuvers and low thrust maneuvers.

Identification of the non-evolutionary change of the orbit requires rather good knowledge of the satellite orbital motion, which can be acquired by processing of the measurements obtained by the sensors. This paper assumes that the satellite orbits are already determined before the maneuver and after. In case the measurements were continuous we could consider only one burn and low thrust maneuvers. The techniques described in this work use a number of classical problems. The paper describes the techniques for solving them with adjustments required by the goal task.

The problem of the determination of the orbital transfers for given initial and final orbits has an infinite number of solutions. Selection of the most probable scheme of maneuvering is based on the evaluation of the characteristic velocity. Selection of one of three (one burn, two burn or low thrust) variants of maneuvering is based on the following criteria:

We assume a one burn transfer when the orbits before and after the maneuver intersect and the value of the characteristic velocity does not exceed a specified limit.

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The two burn transfer is always possible. The times of the burns are selected by the condition of the minimum of the characteristic velocity. The evaluation of the two burn maneuver can be performed using two techniques: by using the matrices of the partial derivatives or by using the Lambert problem depending on the character of orbital motion.

The low thrust maneuver is evaluated in case we know the spacecraft is equipped with a low thrust engine.

The first section of the report considers the mathematical basis of the algorithms for characterization of the maneuvers. We consider the classical problem of orbit determination given two positions and the time for the transfer between them – the Lambert's problem and the problem of evaluation of the required characteristic velocity for coplanar transfers. The algorithm for the Lambert's problem is based on the techniques suggested by the Russian scientist Subbotin[1,2,3]. The geometric method is used for evaluation of energy supply required for coplanar maneuvers.

The second section of the report considers the evaluation of the impulse of the burn when we know the orbital parameters before and after the burn. In the case of a one burn maneuver the tracks before and after the burn must intersect in the point where the burn was applied. Thus the analysis of the cause of the change of the orbital motion should include determination of the point of minimum distance between the tracks before and after the burn. The resulting vector of residuals between orbital positions before and after the burn should be compared with the error of position prediction based on orbital parameters before and after the burn.

The third section of the report considers the case when the parameters of two orbits are known and two burns perform the transfer between them. The algorithm for determination of the times of performing the burns and evaluation of the respective characteristic velocities is described.

The fourth section describes the algorithm for determination the time interval for the work of low thrust engine and the generated acceleration providing the required orbital transfer.

MATHEMATICAL BACKGROUND

Development of the algorithms for evaluation of orbital maneuvers requires several classical methods. Description of these methods is given by Vallado [4]. In particular this book includes a description of the methods of orbit determination for two given positions and the time interval for orbital transfer (Lambert Problem) and the analysis of coplanar maneuvers. This section of the paper considers these classical problems. The algorithm for Lambert's problem is based on the techniques suggested by Subbotin[1,2,3]. The geometric method is used for evaluation of energy supply required for coplanar maneuvers.

Orbit determination for two given positions of the spacecraft

The evaluation of the two burn transfer between orbits 1 and 2 requires analysis of the set of transfer orbits with further selection of the orbit providing the minimum of supplied characteristic velocity. Each transfer orbit is determined by the positions in orbits 1 and 2 and the respective times for these positions. In the scope of the non-perturbed motion the problem of orbit determination for two positions and the time of the transfer between them is a classical Lambert problem.

Theorem (Euler-Lambert)

For the two vectors $\mathbf{r}_1,\ \mathbf{r}_2$ of positions of a mass point in the central gravitational field and the time interval Δt for transfer from position $\mathbf{r}_1$ to position $\mathbf{r}_2$, the following equation is valid:

$$\sqrt{\mu}\cdot\Delta t\cdot a^{-\frac{3}{2}}=\varepsilon-\sin\varepsilon\mp\left(\delta-\sin\delta\right)+2\pi n \tag{1}$$

where

a	–	semi-major axis of the elliptical transfer orbit,
μ	–	gravitational parameter,
n	–	the number of revolutions of the transfer orbit

$$\sin^2\frac{\varepsilon}{2}=\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_2-\mathbf{r}_1|}{4a},\quad \sin^2\frac{\delta}{2}=\frac{|\mathbf{r}_1|+|\mathbf{r}_2|-|\mathbf{r}_2-\mathbf{r}_1|}{4a},$$
$$0\le\frac{\varepsilon}{2}\le\pi,\ -\frac{\pi}{2}\le\frac{\delta}{2}\le\frac{\pi}{2}. \tag{2}$$

$\Delta\upsilon_n=\upsilon_2^n-\upsilon_1^n=\Delta\upsilon+2\pi n,\ 0\le\Delta\upsilon<2\pi$ is the arc passed by the mass point, the sign "–" corresponds to the case when $\Delta\upsilon$ does not exceed π, and the sign "+" to the case when $\Delta\upsilon$ exceeds π. υ_1^n and υ_2^n are true anomalies corresponding to vectors $\mathbf{r}_1,\ \mathbf{r}_2$ taking into account the revolution number.

Euler – Lambert theorem shows that the time for the orbital transfer of the mass point from one position to another depends only on the sum of the radii of these positions $|\mathbf{r}_1|+|\mathbf{r}_2|$, on the value of the chord connecting them $|\mathbf{r}_2-\mathbf{r}_1|$, and on the semi-major axis of the orbit a. Equation (1) is also called Euler-Lambert's equation.

For historical reference[1] we note that this theorem for the parabolic orbit was first proved and published by Euler in 1743[5]. Lambert's theorem, generalizing Euler's result was found in 1761[6].

Albouy[7] noticed that the Euler- Lambert's equation might have solutions not corresponding to any transfer orbits. He suggested the following formulation of Lambert's theorem: Four functions a, $|\mathbf{r}_1|+|\mathbf{r}_2|$, $|\mathbf{r}_2-\mathbf{r}_1|$ and Δt are functionally dependent. Albouy gives references to different proofs of Lambert's theorem and techniques of orbit determination for two positions and the time of transfer: Laplace[8], Adams[9], Dziobek[10], Routh[11], Plummer[12], Battin[13]. Albouy's pedantry provides a platform for the applied algorithm for solving Euler-Lambert equation. We should search for all the solutions of the Euler-Lambert equation. The found solutions should be checked for compliance with the scheme of the transfer from the point $\mathbf{r}_1$ to the point $\mathbf{r}_2$ for the time interval Δt. It should be noted that Albouy considers the cases when the transfer from $\mathbf{r}_1$ to $\mathbf{r}_2$ takes not more than one revolution. The evaluation of the number of solutions for multi-revolution transfers is given in Ref. 14..

Consider the algorithm for determining the transfer trajectory for two given positions and the time of the transfer under the condition that the transfer is performed by not more than a given number of revolutions. This algorithm is based on the methods described in the works[1,2,3] and a note of Albouy[7]. All the solutions of Eq. (1) are searched for. Then the found solutions are checked for compliance with the scheme of the transfer from point $\mathbf{r}_1$ to the point $\mathbf{r}_2$ for time interval Δt.

The design of the algorithm uses the following facts, 1,2,3:

– we always have: $\sin\frac{\varepsilon}{2} > 0$, $\cos\frac{\delta}{2} > 0$;

the sign of $\sin\frac{\delta}{2}$ is determined by the sign of $\cos\frac{\Delta\upsilon_n}{2}$, i.e.

$$\operatorname{sign}\left(\sin\left(\frac{\delta}{2}\right)\right) = \begin{cases} 1, & 0 \le \Delta\upsilon_n < \pi \\ -1, & \pi \le \Delta\upsilon_n < 2\pi \end{cases} \tag{3}$$

$\cos\frac{\varepsilon}{2} < 0$, when the second focus of the transfer ellipse is within the elliptical sector corresponding to the transfer trajectory.

Let us introduce the following notation:

$$\sin\frac{\varepsilon_0}{2} = \sqrt{\frac{|\mathbf{r}_1| + |\mathbf{r}_2| + |\mathbf{r}_2 - \mathbf{r}_1|}{4a}}, \quad \sin\frac{\delta_0}{2} = \sqrt{\frac{|\mathbf{r}_1| + |\mathbf{r}_2| - |\mathbf{r}_2 - \mathbf{r}_1|}{4a}}, \\ 0 \le \frac{\varepsilon_0}{2} \le \frac{\pi}{2}, \quad 0 \le \frac{\delta_0}{2} \le \frac{\pi}{2}. \tag{4}$$

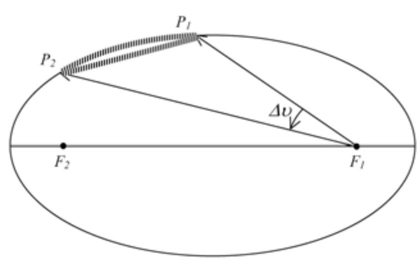


Figure 1. Segment of the Sector Does Not Include Any Foci.

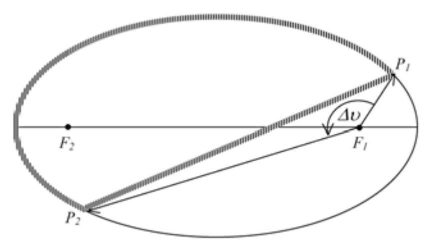


Figure 2. Segment of the Sector Includes the Second Focus $\Delta\upsilon < \pi$

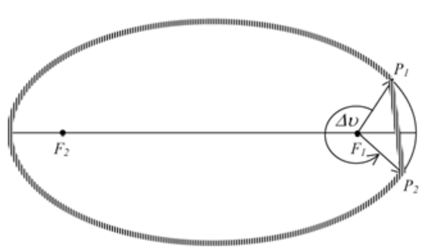


Figure 3. Segment of the Sector Includes the Second Focus $\Delta\upsilon > \pi$

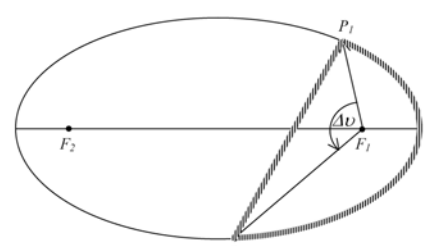


Figure 4. Segment of the Sector Includes Only the First Focus

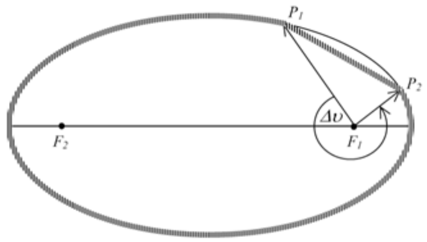


Figure 5. Segment of the Sector Includes Both Foci

Now consider the following possible positions of the elliptical sector corresponding to the transfer trajectory:

– there are no foci within the elliptical sector, case A, Figure 1,
– elliptical sector covers only the second focus, case B, Figures 2, 3,
– elliptical sector covers only the first focus, case C, Figure 4,
– elliptical sector covers both foci, case D, Figure 5.

In the case A $\varepsilon = \varepsilon_0,\ \delta = \delta_0$. In the case B $\varepsilon = 2\pi - \varepsilon_0,\ \delta = \delta_0$ or $\delta = -\delta_0$. In the case C $\varepsilon = \varepsilon_0,\ \delta = \delta_0$ or $\delta = -\delta_0$. In the case D $\varepsilon = 2\pi - \varepsilon_0,\ \delta = -\delta_0$.

Thus, instead of Eq. (1) we can consider four equations:

$$\Delta t = \frac{1}{\sqrt{\mu}} a^{\frac{3}{2}} \left(\varepsilon_0 - \sin\varepsilon_0 - \left(\delta_0 - \sin\delta_0 \right) + 2\pi n \right) \tag{5}$$

$$\Delta t = \frac{1}{\sqrt{\mu}} a^{\frac{3}{2}} \left(2\pi - \left(\varepsilon_0 - \sin\varepsilon_0 \right) - \left(\delta_0 - \sin\delta_0 \right) + 2\pi n \right) \tag{6}$$

$$\Delta t = \frac{1}{\sqrt{\mu}} a^{\frac{3}{2}} \left(\varepsilon_0 - \sin\varepsilon_0 + \left(\delta_0 - \sin\delta_0 \right) + 2\pi n \right) \tag{7}$$

$$\Delta t = \frac{1}{\sqrt{\mu}} a^{\frac{3}{2}} \left(2\pi - \left(\varepsilon_0 - \sin\varepsilon_0 \right) + \left(\delta_0 - \sin\delta_0 \right) + 2\pi n \right) \tag{8}$$

Among the solutions of these equations we may find some that do not correspond to transfer trajectories, i.e. those that do not start and finish in the given points. Thus we will call these four equations an extension of the Euler-Lambert equation. The solutions of the extended Euler-Lambert equation should be checked for compliance with the transfer trajectories.

The selection of the numerical method for solving the equations should account for the character of the functional dependence of the right parts of the equations (5)-(8) on a. The functions of the right parts of the equations (5) and (7), monotonically decrease with the increase of a for $n = 0$ and monotonically increase for $n = 1,\ 2,\ \dots$. The functions of the right parts of the equations (6) and (8), increase for all values of $n = 0,\ 1,\ 2,\ \dots$. Thus for solving equations (5)-(8) we can use the bisection method and the golden section method.

The algorithm described further was used for the task of recovery of the orbital injection scheme for the case of injection with the change of inclination[14].

Algorithm for Lambert problem

Input information:

$\mathbf{r}_{src}$	–	initial position of the spacecraft;
$\mathbf{r}_{trg}$	–	final position of the spacecraft;
Δt	–	time interval for the transfer;
$N_{\max}$	–	maximum number of revolutions for the transfer.

Output information:

N_{ELM}	–	the number of acquired solutions (the number of records in the output array);
$\left\{\Omega_k, i_k, \omega_k, e_k, p_k, t_{\Pi,k}\right\}$, $k = 1,\ldots,N_{ELM}$	–	Output array, each record contain the parameters of the transfer orbit:

Where

Ω_k	–	longitude of ascending node;
i_k	–	inclination;
ω_k	–	pericenter argument;
e_k	–	eccentricity;
p_k	–	parameter of elliptical orbit;
$t_{P,k}$	–	time of passing the pericenter.

Description of the algorithm

1. If the vectors $\mathbf{r}_{src}$ and $\mathbf{r}_{trg}$ are collinear, the algorithm is finished with a negative return code. Otherwise we calculate the vector $\mathbf{m}^0 = \begin{pmatrix} m_x^0, & m_y^0, & m_z^0 \end{pmatrix}^T = \dfrac{\mathbf{r}_{src} \times \mathbf{r}_{trg}}{\left|\mathbf{r}_{src} \times \mathbf{r}_{trg}\right|}$.

2. Calculate 2 vectors: $\mathbf{c}^{01} = \mathbf{m}^0 \cdot sign\left(m_z^0\right)$ and $\mathbf{c}^{02} = -\mathbf{m}^0 \cdot sign\left(m_z^0\right)$,

where $sign\left(m_z^0\right)$ is the sign of the z-component of vector $\mathbf{m}^0$.

The index k of the resulting array is set to zero.

Items 3-5 are performed for each vector $\mathbf{c}^{01}$ and $\mathbf{c}^{02}$. In the description of the algorithm these vectors will be denoted as $\mathbf{c}^{0i}$, $i = 1,2$. The values calculated on their basis will have index i .as well.

3. Then calculate the inclination: $i_i = \arccos\left(c_z^{0i}\right)$, where c_z^{0i} - is the z-component of vector $\mathbf{c}^{0i}$

4. The longitude of the ascending node Ω_i is determined using conditions:

$$\sin\Omega_i = \frac{c_x^{0i}}{\sin i_i}, \ \cos\Omega_i = \frac{c_y^{0i}}{\sin i_i} \tag{9}$$

5. Calculate the difference of true anomalies $\Delta\upsilon_i$ of two given positions using the values $\sin\Delta\upsilon_i$ and $\cos\Delta\upsilon_i$, calculated by formulas:

$$\cos\Delta\upsilon_i = \frac{\left(\mathbf{r}_{src}, \mathbf{r}_{trg}\right)}{\left|\mathbf{r}_{src}\right|\left|\mathbf{r}_{trg}\right|}, \ \sin\Delta\upsilon_i = \left(\mathbf{c}^{0i}, \mathbf{m}^0\right)\frac{\left|\mathbf{r}_{src} \times \mathbf{r}_{trg}\right|}{\left|\mathbf{r}_{src}\right|\left|\mathbf{r}_{trg}\right|}. \tag{10}$$

The cycle for index i is completed. The yield is the two values of the difference of the true anomalies: $\Delta\upsilon_1$ and $\Delta\upsilon_2$.

6. Then calculate the length s of the interval connecting the initial $\mathbf{r}_{src}$ and final $\mathbf{r}_{trg}$ positions:

$$.\, s = \left|\mathbf{r}_{src} - \mathbf{r}_{trg}\right|. \tag{11}$$

7. Using the algorithm for solving the extended Lambert's equation we find two arrays of solutions:

$$\begin{aligned} &\left\{a_{1,k}, \varepsilon_{1,k}, \delta_{1,k}\right\}, \ k = 1,\dots,N_{1,LEQ} \\ &\left\{a_{2,k}, \varepsilon_{2,k}, \delta_{2,k}\right\}, \ k = 1,\dots,N_{2,LEQ} \end{aligned} \tag{12}$$

The first array corresponds to the value of the difference of true anomalies $\Delta\upsilon_1$, and the second one - to $\Delta\upsilon_2$.

8. Then we calculate the unit vector corresponding to vector $\mathbf{r}_{src}$

$$\mathbf{r}_{src}^0 = \begin{pmatrix} x_{src}^0, & y_{src}^0, & z_{src}^0 \end{pmatrix}^T = \frac{\mathbf{r}_{src}}{\left|\mathbf{r}_{src}\right|} \tag{13}$$

9. The cycle for $j = 1,\dots,N_{1,LEQ} + N_{2,LEQ}$ performs operations of items 10-16.

10. Five values are generated:

$$i_j = \begin{cases} i_1, \ \text{if} \ j \le N_{1,LEQ} \\ i_2, \ \text{if} \ j > N_{1,LEQ} \end{cases} \qquad \Omega_j = \begin{cases} \Omega_1, \ \text{if} \ j \le N_{1,LEQ} \\ \Omega_2, \ \text{if} \ j > N_{1,LEQ} \end{cases} \qquad a_j = \begin{cases} a_{1,j}, & \text{if} \ j \le N_{1,LEQ} \\ a_{2,j-N_{1,LEQ}+1}, & \text{if} \ j > N_{1,LEQ} \end{cases}$$

$$\varepsilon_j = \begin{cases} \varepsilon_{1,j} & \text{, if } j \le N_{1,LEQ} \\ \varepsilon_{2,j-N_{1,LEQ}+1}, & \text{if } j > N_{1,LEQ} \end{cases} \qquad \delta_j = \begin{cases} \delta_{1,j} & \text{, if } j \le N_{1,LEQ} \\ \delta_{2,j-N_{1,LEQ}+1}, & \text{if } j > N_{1,LEQ} \end{cases}$$

11. Then calculate the argument of latitude:

$$u = \begin{cases} \arccos(x^0_{src}\cos\Omega_j + y^0_{src}\sin\Omega_j), & \text{if } z^0_{src} \ge 0 \\ 2\pi - \arccos(x^0_{src}\cos\Omega_j + y^0_{src}\sin\Omega_j), & \text{if } z^0_{src} < 0 \end{cases} \tag{14}$$

12. Then we calculate the parameters connecting the auxiliary angles ε and δ with the values of eccentric anomaly in the initial and final points.

$$E_2 - E_1 = E_{2m1} = \frac{\varepsilon_j - \delta_j}{2} \tag{15}$$

$$e\cos E_{2p1} = e_{cE2p1} = \left(1 - \frac{|\mathbf{r}_{src}| + |\mathbf{r}_{trg}|}{2a_j}\right) \cdot \frac{1}{\cos E_{2m1}} \tag{16}$$

$$e\sin E_{2p1} = e_{sE2p1} = \frac{|\mathbf{r}_{src}| - |\mathbf{r}_{trg}|}{2a_j} \cdot \frac{1}{\sin E_{2m1}}\,, \quad E_{2p1} = \frac{E_1 + E_2}{2} \tag{17}$$

Using the values of the sine and cosine of the half-sum of eccentric anomalies we can calculate the value of this half-sum E_{2p1}. For this purpose we can use the function **atan2( ).**

The values of the eccentric anomalies for the initial E_1 and the final E_2 points are calculated using formulas:

$$E_1 = E_{2p1} - E_{2m1} \tag{18}$$

$$E_2 = E_{2p1} + E_{2m1} \tag{19}$$

13. Then we calculate: the value of the eccentricity

$$e_j = \frac{1 - \dfrac{|\mathbf{r}_{src}| + |\mathbf{r}_{trg}|}{2a}}{\cos\dfrac{E_1 - E_2}{2} \cdot \cos\dfrac{E_1 + E_2}{2}} \tag{20}$$

14. the true anomaly for the initial point:

$$\upsilon = 2\,\mathrm{arctg}\left(\sqrt{\frac{1+e_j}{1-e_j}}\,\mathrm{tg}\frac{E_1}{2}\right) \tag{21}$$

15. the argument of the pericenter:

$$\omega_j = u - \upsilon \tag{22}$$

16. time of passing the pericenter:

$$t_{P,j} = -\sqrt{\frac{a_j^3}{\mu}}\left(E_1 - e\sin E_1\right) \tag{23}$$

17. parameter of the elliptical orbit:

$$p_j = a_j\left(1 - e_j^2\right) \tag{24}$$

18. Using the determined elements we calculate the state vectors for the times $t_1 = 0$ and $t_2 = \Delta t$: $\mathbf{r}_{j,1}$ and $\mathbf{r}_{j,2}$. Then we calculate the residual: $d_j = \left|\mathbf{r}_{src} - \mathbf{r}_{j,1}\right| + \left|\mathbf{r}_{trg} - \mathbf{r}_{j,2}\right|$. If d_j is smaller than the set threshold value the solution is saved, otherwise - discarded. In case the solution is saved we make the following assignments:

$k = k+1,\ \Omega_k = \Omega_j,\ i_k = i_j,\ \omega_k = \omega_j,\ e_k = e_j,\ p_k = p_j,\ t_{P,k} = t_{P,j}$

19. After the completion of the cycle for j index k is used for forming the length of the output array: $N_{ELM} = k$.

Algorithm for the Extended Lambert Equation

The algorithm finds the solutions of the extended Lambert's equation (1). The algorithm generates the array containing the values of the semi-major axis and auxiliary angles ε and δ . The true solutions of Lambert's equation, i.e. the solutions corresponding to certain transfer orbit are among the elements of this array. The search for the true solutions requires an additional check which is performed by the algorithm of higher level.

Input information:

$\left\|\mathbf{r}_{src}\right\|$	–	Magnitude of the vector defining the initial position of the spacecraft,
$\left\|\mathbf{r}_{trg}\right\|$	–	Magnitude of the vector defining the final position of the spacecraft,
Δt	–	time interval of the transfer,
$N_{\max}$	–	maximum of the number of revolutions for the transfer,
$\Delta\vartheta$	–	difference of true anomalies.

Output data:

N_{LEQ}	–	the number of elements in the array,
$\left\{a_k, \varepsilon_k, \delta_k, N_k\right\}\ \ k = 1,\ldots,N_{LEQ}$	–	determined values.

We set the initial value of index $k = 0$.

The operations 1-3 are performed in the cycle for N from 0 to $\left(N_{\max}-1\right)$. We calculate the minimum possible value of the semi-major axis $a=\frac{\left|\mathbf{r}_{src}\right|+\left|\mathbf{r}_{trg}\right|+s}{4}$ of the transfer trajectory and the angle $0<\delta_0<\pi$, determined by the equality:

$$\sin\frac{\delta_0}{2}=\sqrt{\frac{\left|\mathbf{r}_{src}\right|+\left|\mathbf{r}_{trg}\right|-s}{4a}} \tag{25}$$

The minimum value of the semi-major axis can be attained only for the case boundary elliptical sector. In this case the segment connecting the ends of radius- vectors r_1 and r_2, includes the second focus of the ellipse, the angle $\varepsilon=\pi$.

If $\Delta\vartheta\leq\pi$, we check the equality:

$$\sqrt{\mu}\Delta t=a^{\frac{3}{2}}\left[\pi-\left(\delta_0-\sin\delta_0\right)+2\pi N\right] \tag{26}$$

If the equality is satisfied we save in the output array the values: $a,\ \varepsilon=\pi, \delta=\delta_0, N$. For doing this we add 1 to the index k and make the assignments: $a_k=a,\ \varepsilon_k=\pi,\ \delta_k=\delta_0,\ N_k=N$. After that we return to the beginning of the cycle with new value of N.

If $\pi<\Delta\vartheta\leq 2\pi$, we check the equality:

$$\sqrt{\mu}\Delta t=a^{\frac{3}{2}}\left[\pi+\left(\delta_0-\sin\delta_0\right)+2\pi N\right] \tag{27}$$

If the equality is satisfied we make the assignments: $k=k+1$, $a_k=a$, $\varepsilon_k=\pi$, $\delta_k=-\delta_0$, $N=N_k$ and we go to the beginning of the cycle with new value of N.

If none of the above mentioned equalities are satisfied we go to the next item 2.

1. Within the interval from $a_{\min}=\frac{\left|\mathbf{r}_{src}\right|+\left|\mathbf{r}_{trg}\right|+s}{4}+\sigma_a$ to $a_{\max}$ we search for the zeroes of two functions whose shape depends on the value of the angle $\Delta\vartheta$. Here the parameter σ_a is of the order of unit meters, and $a_{\max}$ does not exceed 300,000 km.

If $\Delta\vartheta\leq\pi$, we search for the zeroes of functions:

$$\begin{gathered}\Phi_1\left(a\right)=\sqrt{\mu}\Delta t-a^{\frac{3}{2}}\left[\varepsilon_0-\sin\varepsilon_0-\left(\delta_0-\sin\delta_0\right)+2\pi N\right],\\ \Phi_2\left(a\right)=\sqrt{\mu}\Delta t-a^{\frac{3}{2}}\left[2\pi-\left(\varepsilon_0-\sin\varepsilon_0\right)-\left(\delta_0-\sin\delta_0\right)+2\pi N\right].\end{gathered} \tag{28}$$

If $\pi<\Delta\vartheta\leq 2\pi$, we search for the zeroes of functions:

$$\Phi_3(a)=\sqrt{\mu}\Delta t-a^{\frac{3}{2}}\left[\varepsilon_0-\sin\varepsilon_0+(\delta_0-\sin\delta_0)+2\pi N\right],$$
$$\Phi_4(a)=\sqrt{\mu}\Delta t-a^{\frac{3}{2}}\left[2\pi-(\varepsilon_0-\sin\varepsilon_0)+(\delta_0-\sin\delta_0)+2\pi N\right]. \tag{29}$$

Where

$$\sin\frac{\varepsilon_0}{2}=\sqrt{\frac{|\mathbf{r}_{src}|+|\mathbf{r}_{trg}|+s}{4a}}\ ,\ \sin\frac{\delta_0}{2}=\sqrt{\frac{|\mathbf{r}_{src}|+|\mathbf{r}_{trg}|-s}{4a}}\ . \tag{30}$$

Thus, using the bisection algorithm within the interval $\left[a_{\min}, a_{\max}\right]$ we find the value $a_{\Phi=0}$, for which the function Φ_k is equal to zero.

3. Then we calculate the angles $0<\varepsilon_0<\pi$, $0<\delta_0<\pi$, determined by the equalities:

$$\sin\frac{\varepsilon_0}{2}=\sqrt{\frac{|\mathbf{r}_{src}|+|\mathbf{r}_{trg}|+s}{4a_{\Phi=0}}}\ ,\ \sin\frac{\delta_0}{2}=\sqrt{\frac{|\mathbf{r}_{src}|+|\mathbf{r}_{trg}|-s}{4a_{\Phi=0}}}\ . \tag{31}$$

Depending on the number of the function k we calculate the angles ε and δ :

k	ε	δ
1	$\varepsilon=\varepsilon_0$	$\delta=\delta_0$
2	$2\pi-\varepsilon_0$	$\delta=\delta_0$
3	$\varepsilon=\varepsilon_0$	$\delta=-\delta_0$
4	$2\pi-\varepsilon_0$	$\delta=-\delta_0$

In the output array we save the found value of the semi-major axis $a_{\Phi=0}$ and the corresponding values of ε, δ, N . For this we make the assignments: $k=k+1$, $a_k=a_{\Phi=0}$, $\varepsilon_k=\varepsilon$, $\delta_k=\delta$, $N_k=N$.

4. After completion of the cycle the index k is used for forming the length of the output array: $N_{LEQ}=k$.

Evaluation of the energy costs for coplanar transfers. Geometrical method

For maneuver characterization problems we always know the state vectors for the spacecraft before and after the maneuver. In the scope of spacecraft control theory these problems are called rendezvous problems. Thus we should use the techniques for solving rendezvous problem as the basis for the maneuver characterization problem. In the case of the non- perturbed motion this is the Lambert problem. However, for evaluation of the results we should compare the obtained energy costs with the case for which we have the parameters of the target orbit, but the position in this orbit is not defined. When we control the spacecraft for reaching certain position for a given time we perform phasing maneuvers which provides such conditions for the orbital transfer that the energy cost for the orbital transfer with the set position in the target orbit does not significantly differ from the orbital transfer with an arbitrary position of the spacecraft in the target orbit after the transfer maneuver.

Let us consider the geometrical method of the transfer from one coplanar orbit to another. Consider the family of ellipses with the focus in the origin of coordinates. This family of ellipses is described by the parameters l and f which are determined by the following relationships

$$c = \frac{|f|}{2}, \; a = \frac{l}{2}, \; e = \frac{c}{a} = \frac{|f|}{a}, \tag{32}$$

where
- c – half distance between the foci
- a – semi-major axis
- e – eccentricity

If the pericenter is placed to the left of the coordinate origin then $f > 0$, otherwise $f < 0$.

The distance to the spacecraft and the velocity at apocenter r_α, V_α and pericenter r_π, V_π have the following relationships with l and f:

$$\begin{aligned} r_\alpha &= \frac{l+|f|}{2}, \quad V_\alpha = \sqrt{\frac{2\mu}{l} \cdot \frac{l-|f|}{l+|f|}}, \\ r_\pi &= \frac{l-|f|}{2}, \quad V_\pi = \sqrt{\frac{2\mu}{l} \cdot \frac{l+|f|}{l-|f|}}, \end{aligned} \tag{33}$$

where μ is the gravitational constant.

Each orbit is represented by a point in the semi-plane l, f, $l > 0$. If $f > 0$, we have apocenter right from the origin of coordinates, if $f < 0$ - pericenter. We will consider applying the burn pulses only at pericenter and apocenter.

Let us consider the orbital transfer from the point l_1, f_1 to the point l_2, f_2. If the burn is applied at the apsidal point right-side from the origin of coordinates, the orbital transfer must keep the distance to this apsidal point. For $f_1 > 0, f_2 > 0$ - this is an orbital transfer for which the apocenter will remain right-side from the origin of coordinates before and after the burn, thus

$$\frac{l_1 + f_1}{2} = \frac{l_2 + f_2}{2} \tag{34}$$

The case $f_1 > 0$, $f_2 < 0$ correspond to the orbital transfer for which we will have the pericenter right-side from the origin of coordinates after the burn is applied. Thus:

$$\frac{l_1 + f_1}{2} = \frac{l_2 - |f_2|}{2} = \frac{l_2 + f_2}{2} \tag{35}$$

Consideration of the cases $f_1 \le 0$, $f_2 < 0$ and $f_1 \le 0$, $f_2 > 0$, yields that the application of the burn on the right-side of the origin of coordinates keeps the value $l + f$.

Similar considerations on the cases when the burn is applied left-side from the origin of coordinates will result in the conclusion that for these cases the value $l+f$ is kept. Thus the broken line in the semi-plane $l,\ f,\ l>0$, corresponding to the sequence of orbital transfers will consist of segments with inclination angle tangents -1 and +1 (Fig. 6). The value -1 for the inclination angle tangent corresponds to the burn applied right-side from the origin of coordinates and the value +1 – corresponds to the left-side application of burn. The velocities in the apsidal points (in the apocenter or pericenter) right-side V_r and left-side V_l from the origin of coordinates are calculated using formulas:

$$V_r = \sqrt{\frac{2\mu(l-f)}{l(l+f)}},\ V_l = \sqrt{\frac{2\mu(l+f)}{l(l-f)}}. \tag{36}$$

$$\Delta V_r = \left|\sqrt{\frac{2\mu(l_2-f_2)}{l_2(l_2+f_2)}} - \sqrt{\frac{2\mu(l_1-f_1)}{l_1(l_1+f_1)}}\right|,\ \Delta V_l = \left|\sqrt{\frac{2\mu(l_2+f_2)}{l_2(l_2-f_2)}} - \sqrt{\frac{2\mu(l_1+f_1)}{l_2(l_1-f_1)}}\right|. \tag{37}$$

For the orbital transfer from the point $l_1,\ f_1$ to the point $l_2,\ f_2$ the values of the transfer impulses ΔV_r (when the burn is applied right-side from the origin of coordinates) and ΔV_l (when the burn is applied left-side from the origin of coordinates) are calculated using the formulas:

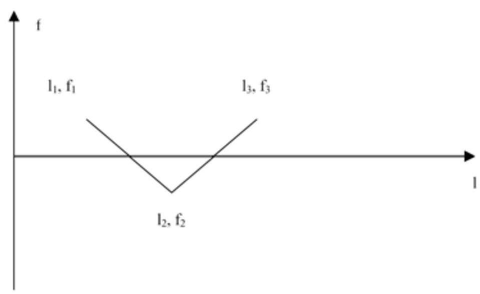


Figure 6. Orbital Transfers Depicted In the Semi-Plane $l,\ f,\ l>0$. For the Transfer From the Point $l_1,\ f_1$ To the Point $l_2,\ f_2$ The Burn is Applied Right-Side From the Origin Of Coordinates, For the Transfer From $l_2,\ f_2$ To the Point $l_3,\ f_3$ – To the Left Side

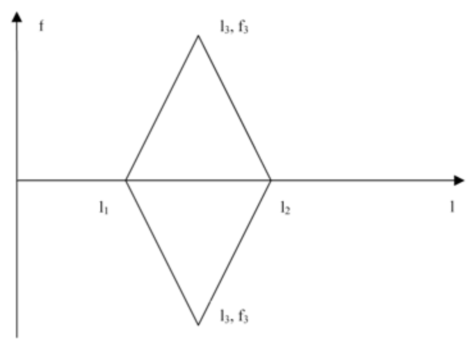


Figure 7. Hohmann's Transfer Plotted In the Semi-Plane $l,\ f,\ l>0$. The Upper Broken Line Corresponds To the First Burn Applied Left-Side From the Origin of Coordinates, the Lower Line – To the Right-Side Application.

Let us consider the Hohmann's transfer from the circular orbit with radius l_1 to the circular orbit with radius l_2 (Figure 7). The elliptical transfer orbit will have the parameters: $l_3 = \frac{l_1+l_2}{2},\quad f_3 = \pm\frac{l_2-l_1}{2}$. The sign «+» correspond to the burn applied to the left-side of the origin of coordinates and the sign «-» - to the right-side application. Using (37) for $l_2 > l_1$, we will obtain the consumption of the characteristic velocity for the Hohmann's transfer:

$$\Delta V_H = \sqrt{\frac{2\mu}{l_1}} C(\rho), \qquad C(\rho) = \left(\sqrt{\frac{2\rho}{1+\rho}} - 1 + \sqrt{\frac{1}{\rho}} - \sqrt{\frac{2}{\rho(\rho+1)}} \right), \tag{38}$$

where $\rho = \frac{l_2}{l_1}$.

The graph of the function $C(\rho)$ is shown in Figure 8.

Let us consider the three burn bi-elliptical transfer from the circular orbit with radius l_1 to the circular orbit with radius l_2 (Figure 9). The first burn is applied left-side from the origin of coordinates and makes the semi-major axis of the first transfer orbit α times greater, i.e. $l_{p1} = \alpha l_1$, where $\alpha > 1$. Parameters of intermediate orbits $l_{p1}, f_{p1}, l_{p2}, f_{p2}$ are connected by the following relationships:

$$\frac{l_1}{2} = \frac{l_{p1} - f_{p1}}{2}, \quad \frac{l_{p1} + f_{p1}}{2} = \frac{l_{p2} + f_{p2}}{2}, \quad \frac{l_{p2} - f_{p2}}{2} = \frac{l_2}{2}. \tag{39}$$

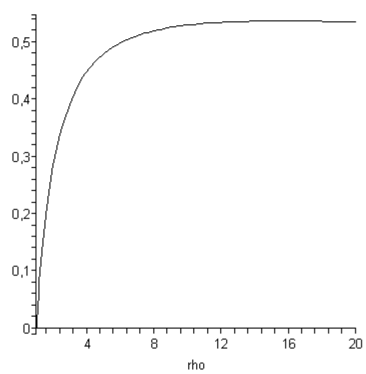


Figure 8. Graph Of the Function $C(\rho)$.

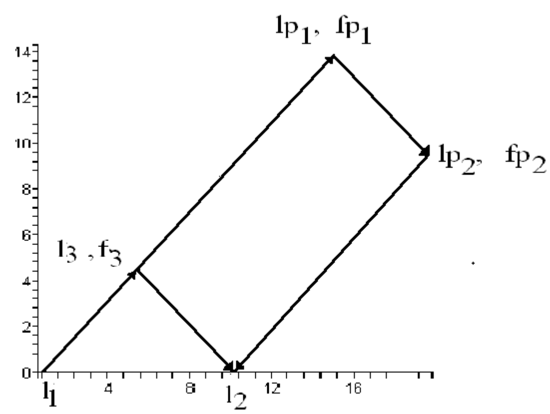


Figure 9. The Three Burn Bi-Elliptical Transfer From the Circular Orbit With Radius l_1 To the Circular Orbit With Radius l_2 and the Hohmann's Transfer. Hohmann's Transfer Corresponds To the Broken Line Including the Points: $(l_1,0)$ -> (l_3,f_3) -> $(l_2,0)$. The Three Burn Bi-Elliptical Transfer Corresponds To the Broken Line Including the Points: $(l_1,0)$ -> (lp_1,fp_1) -> (lp_2,fp_2) -> $(l_2,0)$.

Solving the system of equations (39) with respect to f_{p1}, l_{p2}, f_{p2}, we will have:

$$f_{p1} = (\alpha - 1) l_1,\ l_{p2} = \frac{(2\alpha - 1) l_1 + l_2}{2},\ f_{p2} = \frac{(2\alpha - 1) l_1 - l_2}{2} \tag{40}$$

The burn impulses for the first, the second and the third segments of the bi-elliptical transfer are calculated using formulas:

$$\Delta V_1 = \sqrt{\frac{2\mu}{l_1}} \left(\sqrt{\frac{2\alpha - 1}{\alpha}} - 1 \right),$$

$$\Delta V_2 = \sqrt{\frac{2\mu}{l_1}} \sqrt{\frac{1}{2\alpha - 1}} \left(\sqrt{\frac{2 l_2}{(2\alpha - 1) l_1 + l_2}} - \sqrt{\frac{1}{\alpha}} \right), \tag{41}$$

$$\Delta V_3 = \sqrt{\frac{2\mu}{l_2}} \left(1 - \sqrt{\frac{2(2\alpha - 1) l_1}{(2\alpha - 1) l_1 + l_2}} \right)$$

The total Δv for the three burn bi-elliptical transfer is calculated by the formula:

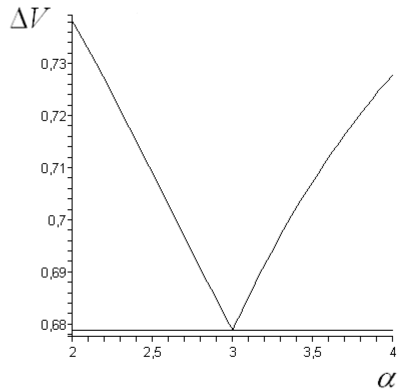
$$\Delta V_{3B} = |\Delta V_1| + |\Delta V_2| + |\Delta V_3| \tag{42}$$

For $\alpha = (l_1 + l_2)/2$ the bi-elliptical three burn transfer degenerates into the Hohmann's transfer and the value of ΔV_{3B} becomes equal to ΔV_H.

The non-dimensional values l_1, l_2 - mean the ratios of the radii of the orbits to mean Earth radius. For transition to the dimensional value of the magnitude of the characteristic velocity we should multiply the non-dimensional value by the value of the first escape velocity $\sqrt{\mu / R_E} = 7.905365716$ km/s.

We will compare the characteristic velocity for the three burn bi-elliptical transfer with that for the Hohmann's transfer. Figure 10 shows the dependence of ΔV_{3B} from α for $l_1 = 1, l_2 = 2$ with the characteristic velocity for the Hohmann's transfer at the background which are presented by the straight line parallel to the abscissa axis. For $\alpha = 1.5$ the three burn transfer degenerates into two burn Hohmann's transfer with energy consumption of 0.67884128 in non-dimensional values. For the values of α greater or lower than 1.5 the consumption for the three burn transfer is greater than that for the Hohmann.

Let us now set $\alpha = 1 + l_2$ and compare the characteristic velocity for the three burn and Hohmann's transfer from the circular orbit with radius $l_1 = 1$ to the orbit with radius l_2. Figure 11 shows the difference of the characteristic velocity for the three burn and Hohmann's transfer. Within the interval of l_2 values from 13 to 14 the difference in the characteristic velocity changes from 0.0029183931 to -0.0006411714, i.e., it changes sign. Thus for the value of $l_2 = 14$ we have less change in characteristic velocity for the three burn bi-elliptical transfer than for the Hohmann transfer.



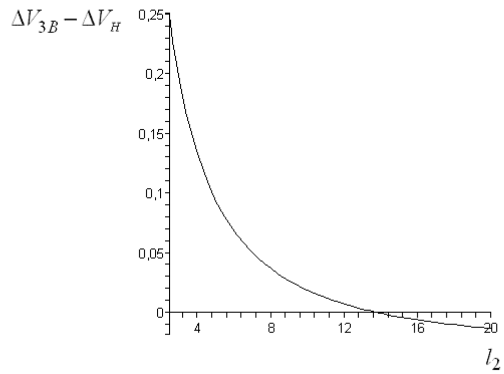


Figure 10. Comparing The Characteristic Velocity For the Three Burn Bi-Elliptical and Hohmann's Transfer For the Radii of 1 And 3 For Initial and The Target Orbit Respectively. The Abscissa Axis Presents α - The Ratio Of the Pericenter Distance of the First Transfer Orbit To the Radius of the Initial Orbit. The Ordinate Axis Presents the Characteristic Velocity Consumption In Non-Dimensional Characteristic Velocity Consumption For the Hohmann's Transfer.

Figure 11. The Difference of Characteristic Velocity (In Non-Dimensional Values). For the Three Burn Bi-Elliptical and Hohmann's Transfer As Function of l_2 for $l_1 = 1,\ \alpha = 1 + l_2$

The general result is known [4]. For the ratio of orbital radiuses $l_2 / l_1 < 11.94$ the characteristic velocity for the Hohmann transfer is smaller than for the three burn bi-elliptical transfer. For the ratio of the radii $11.94 < l_2 / l_1 < 15.58$ there exists the value of α, for which the consumption for the three burn transfer is smaller than for the Hohmann's one. For the ratio of the orbital radii $l_2 / l_1 > 15.58$ the characteristic velocity for the three burn transfer is smaller than for the Hohmann transfer for any value of $\alpha > 1$. However, the increase in the characteristic velocity by using the three burn bi-elliptical transfer instead of Hohmann's one does not exceed 8%, and the time required for the transfer increases several times.

The above considerations lead to the following conclusions. The evaluation of the characteristic velocity for the Hohmann's transfer is the basic value for checking the reliability of other possible maneuvering profiles with regard to energy cost. The geometrical technique for analysis of orbital transfers is an efficient tool for analysis of maneuvering profiles without changes of the orbital plane.

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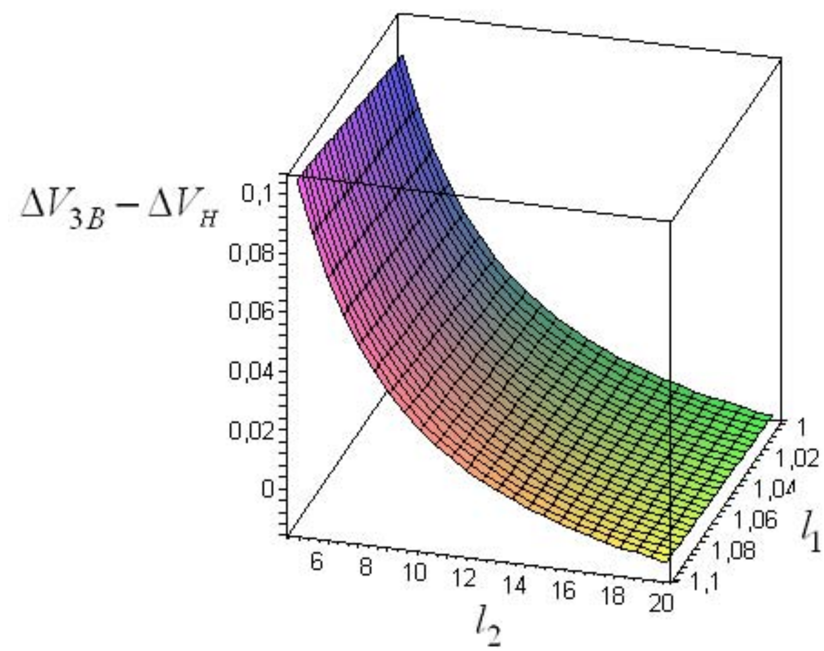

Figure 12. Difference Of Characteristic Velocity Consumption (In Non-Dimensional Values) For the Three Burn Bi-Elliptical and Hohmann's Transfer As Function of l_1 and l_2 for $l_1 = 1$, $\alpha = 1 + l_2$

Figure 12. presents the difference in the characteristic velocity for the three burn and Hohmann's transfers for $1 \le l_1 \le 1.1,\ 5 \le l_2 \le 20,\ \alpha = l_1 + l_2$.

The above considerations lead to the following conclusions. The evaluation of the characteristic velocity for the Hohmann's transfer is the basic value for checking the reliability of other possible maneuvering profiles with regard to energy cost. The geometrical technique for analysis of orbital transfers is an efficient tool for analysis of maneuvering profiles without changes of the orbital plane.

ALGORITHM FOR EVALUATION OF ONE BURN MANEUVER

The change of orbital parameters caused by thrust forces, in the case the orbital parameters (TLE) before and after the burn are available can be interpreted as a one burn maneuver. This is the most frequent case. If the determination of the orbital parameters occurred more frequently than the maneuvers, all the changes in the orbits could be interpreted as one burn maneuvers.

If a one burn maneuver occurred the trajectories before and after the burn must intersect in the point where the thrust was applied. Thus the analysis of possible causes of the change of the orbital parameters we should search for the point corresponding to the minimum distance between the trajectories before and after the thrust. The resulting vector of residuals between orbital positions before and after the burn should be compared with the error of position prediction based on the orbital parameters before and after the burn.

Algorithm for the search of the minimum distance between the orbits

When we search for the minimum distance between the orbits we know the time interval for the search. This interval coincides with the time interval within which the burn was applied. It should be noted that in case we have available the TLE before the thrust and the TLE after the thrust the epochs of these TLEs should not be identified with the interval of maneuvering. The TLE epoch normally refers to the time of nodal crossing and not to the times of the last measurements of the observation interval for which the TLE have been generated. Thus in case we have the TLE, the interval for the search of the minimum distance should be extended by one period to the left from the epoch of TLE before the burn and by one period to the right from the epoch of TLE after the burn.

Let us introduce the notation:

t_1, t_2	–	Boundaries for the search of the minimum distance between the orbits;
$\mathbf{r}_1(t) = \left(x_1(t), \ y_1(t), \ z_1(t) \right)^{\mathrm{T}}$	–	Position vector of the spacecraft based on orbital data before the burn;
$\mathbf{v}_1(t) = \left(V_{x1}(t), \ V_{y1}(t), \ V_{z1}(t) \right)^{\mathrm{T}}$	–	Velocity vector of the spacecraft based on orbital data before the burn;
$\mathbf{r}_2(t) = \left(x_2(t), \ y_2(t), \ z_2(t) \right)^{\mathrm{T}}$	–	Position vector of the spacecraft based on orbital data after the burn;
$\mathbf{v}_2(t) = \left(V_{x2}(t), \ V_{y2}(t), \ V_{z2}(t) \right)^{\mathrm{T}}$	–	Velocity vector of the spacecraft based on orbital data after the burn.

The task is to find the time t, for which the value $\left|\mathbf{r}_2(t) - \mathbf{r}_1(t)\right|$ reaches its minimum.

For the search of the minimum distance we will use the fact that the derivative of the function $\left|\mathbf{r}_2(t) - \mathbf{r}_1(t)\right|$ must change sign. In the case when for the entire interval $\left[t_1, t_2\right]$ the derivative of the considered function does not change sign, the minimum distance corresponds to one of the boundary values of the interval $\left[t_1, t_2\right]$. The derivative of the function $\left|\mathbf{r}_2(t) - \mathbf{r}_1(t)\right|$ is calculated by the function:

$$\frac{d}{dt}\left|\mathbf{r}_2 - \mathbf{r}_1\right| = \frac{(x_2 - x_1)(V_{x2} - V_{x1}) + (y_2 - y_1)(V_{y2} - V_{y1}) + (z_2 - z_1)(V_{z2} - V_{z1})}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \tag{43}$$

The search for the points where the derivative of the function $\left|\mathbf{r}_2(t)-\mathbf{r}_1(t)\right|$ changes sign is performed by scanning of the interval of the search. However, the scanning should be performed not by the time as variable, but using the true anomaly with permanent step h_υ, equal, for example, to 1° or 10°.

Consider the transition from the time of the previous step t_p to the time of the current step of scanning t. For the time t_p using orbital data before the maneuver we calculate the osculating orbital elements:

M_p – mean anomaly;

υ_p – true anomaly;

e_p – eccentricity;

n_p – mean motion.

The transition to the time t of the current step of scanning is performed using formulas:

$$E = 2\,\mathrm{arctg}\sqrt{\frac{1-e_p}{1+e_p}}\,\mathrm{tg}\frac{\upsilon_p + h_\upsilon}{2}$$

$$M = \begin{cases} E - e_p \sin E, & \text{if } \; E - e_p \sin E \geq M_p, \\ E - e_p \sin E + 2\pi, & \text{if } \; E - e_p \sin E < M_p, \end{cases} \tag{44}$$

$$t = t_p + \frac{M - M_p}{n_p}$$

The scanning is finished when $t > t_2$.

For each step of scanning we calculate the derivative of the distance between the orbits:

$$D(t) = \frac{d}{dt}\left|\mathbf{r}_2(t)-\mathbf{r}_1(t)\right| \tag{44}$$

using Eq. (43). If the condition:

$$D(t_p)\cdot D(t) \leq 0 \tag{45}$$

is satisfied the derivative changes its sign within the interval $\left[t_p, t\right]$. In this case for the time $t_m = (t_p + t)/2$ we calculate the value of the derivative of the distance between orbits $D(t_m)$.

If the values $D_p = D(t_p)$, $D_m = D(t_m)$, $D = D(t)$ are monotonic, i.e. $D_p < D_m < D$ or $D_p > D_m > D$ we calculate the time t_0, when the derivative of the distance between orbits is zero, using the formula:

$$t_0 = t_{\rm p} \frac{D_m \cdot D}{(D_p - D_m)\cdot(D_p - D)} + t_{\rm m} \frac{D_p \cdot D}{(D_m - D_p)\cdot(D_m - D)} + t \frac{D_m \cdot D}{(D - D_m)\cdot(D - D_p)}. \quad (46)$$

If the monotonic condition for the values of D_p, D_m, D is not satisfied, we set: $t_0 = t_{\rm m}$.

Calculation of the impulse for the scheme of the one burn maneuver

Further we perform the calculation of the consumption of characteristic velocity for the maneuver and the vectors of relative positions and velocities in the RNB coordinate frame. For this purpose for the time t_0 we calculate the state vectors $\left(x_1, \ y_1, \ z_1, \ Vx_1, \ Vy_1, \ Vz_1 \right)^T$, $\left(x_2, \ y_2, \ z_2, \ Vx_2, \ Vy_2, \ Vz_2 \right)^T$ using initial conditions before and after the maneuver. Then we calculate the consumption of the characteristic velocity for the maneuver using the formula:

$$DV = \sqrt{(V_{\rm x2} - V_{\rm x1})^2 + (V_{\rm y2} - V_{\rm y1})^2 + (V_{\rm z2} - V_{\rm z1})^2} \quad (47)$$

On the basis of the state vector before the maneuver we calculate the matrix $\mathbf{M}_{\mathbf{RNB}}$ of the transformation to the RNB coordinate frame: $\mathbf{M}_{\mathbf{RNB}} = \left[\ \mathbf{e_r}, \ \mathbf{e_n}, \ \mathbf{e_b} \ \right]^{\rm T}$, where

$\mathbf{e_r} = \dfrac{\mathbf{r}_1}{|\mathbf{r_1}|}$ – unit vector directed to the point $\mathbf{r_1} = \left(x_1, \ y_1, \ z_1 \right)^{\rm T}$,

$\mathbf{e_b} = \dfrac{\mathbf{r_1} \times \mathbf{V_1}}{|\mathbf{r_1} \times \mathbf{V_1}|}$ – unit vector orthogonal to the orbital plane, $\mathbf{V}_1 = \left(V_{x1}, \ V_{y1}, \ V_{z1} \right)^{\rm T}$,

$\mathbf{e_n} = \mathbf{e_r} \times \mathbf{e_b}$. –

The vectors of the relative positions and velocities for the RNB coordinate frame are calculated by the formulas:

$$\mathbf{DR}_{\mathbf{RNB}} = \mathbf{M}_{\mathbf{RNB}} \cdot \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \quad (48)$$

$$\mathbf{DV}_{\mathbf{RNB}} = \mathbf{M}_{\mathbf{RNB}} \cdot \begin{bmatrix} V_{x2} - V_{x1} \\ V_{y2} - V_{y1} \\ V_{z2} - V_{z1} \end{bmatrix} \quad (49)$$

Reliability criteria for the obtained evaluation of the impulse

The major criterion to be used for assessment of the reliability of the obtained evaluation of the impulse is the distance between the positions of the spacecraft in the first and the second orbit for the time of minimum distance. This distance is calculated by the formula:

$$DR = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2} \tag{50}$$

If the value DR exceeds the threshold determined by the accuracy of the orbital data we can consider that by the time of minimum distance the orbits do not intersect and the obtained estimate is not reliable.

When DR is smaller than the threshold we perform additional analysis including comparing the value of DV with the characteristic velocity required for the Hohmann transfer or inclination changing maneuver. If the orbital planes before and after the maneuver coincide, a coplanar maneuver has occurred. Energy cost of this maneuver should be comparable with the cost of one of the impulses of the Hohmann transfer. If the orbital plane has changed the value of DV should be compared with the characteristic velocity required for the inclination changing maneuver.

We should note one very important empirical criterion for the impulses with modules not exceeding 1 m/s. If the maximum (in absolute value) component of the vector $\mathbf{DR}_{\mathbf{RNB}}$ is the component corresponding to the direction of vector $\mathbf{N}$, and the maximum (in absolute value) component of the vector $\mathbf{DV}_{\mathbf{RNB}}$ - is the component corresponding to the direction of vector $\mathbf{R}$, the obtained estimate of the maneuver is not reliable and the difference between the orbital parameters is caused by the errors of orbit determination.

Algorithm for the calculation of the length of the impulse

The length of the impulse can be calculated knowing the characteristic velocity, if the mass of the spacecraft before the burn, the thrust and the specific impulse of the engine are known. Under the conditions when these parameters are not available the length of the impulse can be determined on the basis of the following empirical table which presents the acceleration as function of the characteristic velocity. The table is based on the simple consideration that small impulses would not be generated by high-powered engines and the big ones – by the low thrust engines.

Characteristic velocity, m/s	Acceleration, m/s^2
0-5	0.1
5-70	0.2
70-150	0.5
150-1000	2
1000-…	10

Depending on DV - the estimate of the characteristic velocity of the maneuver, the mean acceleration a is determined. The length of the impulse Δt_{imp} is calculated as $\Delta t_{\text{imp}} = \frac{DV}{a}$. The time of the start of the maneuver $t_{\text{beg\_imp}}$ and the time $t_{\text{end\_imp}}$ of its end are calculated using the formulas:

$$t_{\text{beg\_imp}} = t_0 - \frac{\Delta t_{\text{imp}}}{2}, \quad t_{\text{end\_imp}} = t_0 + \frac{\Delta t_{\text{imp}}}{2} \tag{51}$$

where t_0 is the time corresponding to the minimum distance between orbits (the time of the burn in the scope of impulse understanding)

ALGORITHM FOR EVALUATION OF TWO BURN MANEUVER

Assume the orbital parameters for times t_{E1}, t_{E2}, $t_{E1} < t_{E2}$ and the interval of possible maneuvering t_{m1}, t_{m2} are known. The orbit corresponding to the time t_{E1}, will be called the initial one and the orbit corresponding to the time t_{E2} - the target one. The position vector of the spacecraft as a function of time will be called the trajectory. Corresponding to the initial and target orbit we will differ the initial and the target trajectories. If within the interval t_{m1}, t_{m2} we cannot find the time of closest approach of the initial and target trajectories we will consider that between the times t_{m1} and t_{m2} more than one burn has been performed. Then we will consider the algorithm for the evaluation of the impulses and the times of their application under the assumption that a two burn maneuver has been performed resulting in the transfer from the initial orbit to the target one.

First we will consider the variant when the orbital parameters are represented by position $\mathbf{r}_1, \mathbf{r}_2$ and velocity $\mathbf{v}_1, \mathbf{v}_2$ vectors for the times t_{E1}, t_{E2}.

The task of evaluating the impulses and the times of their application is equivalent to the task of determining the transfer orbit between certain positions in the initial and target trajectories respectively. Selecting the times $t_1, t_2 \in \left(t_{m1}, t_{m2}\right), t_1 < t_2$ we can obtain the infinite set of such orbits. The desired orbit should be selected to satisfy the condition of the minimum consumption of characteristic velocity for transfer.

Selection of the time t_1 defines the position of the spacecraft in the initial trajectory and the selection of the time t_2 defines the position of the spacecraft in the target trajectory. Thus for the search of the transfer orbit we have two positions and the respective times. So the search of the transfer orbit can be considered the solving of the boundary problem for the differential equation describing the motion of the spacecraft. Since here we do not deal with the task of spacecraft flight control and consider the task of evaluation of the maneuvers which have been performed already, we can look for approximate estimates. However, we should mention that the obtained approximate estimate is the initial approximation for obtaining the precise solution.

The considered problem is solved according to the following scheme. We scan the set of transfer orbits and look for the orbit corresponding to the minimum change in characteristic velocity. The transfer orbits are generated approximately using two techniques. The technique for generating the transfer orbits is selected depending on the condition on the value of the impulses required for transition to the transfer orbit from the initial one and for transition from the transfer orbit to the target one. We consider the following cases:

– the values of the impulses are smaller than the given threshold and we can use linearity in the vicinity of the reference trajectory;

– the occurred change of the orbital parameters correspond to the change in characteristic velocity exceeding the set threshold.

When we have the opportunity to use the linearity in the vicinity of the reference trajectory then the basic tool for the approximate generation of the transfer orbit is the matrix of partial derivatives of the components of the state vector for the time t_2 with respect to the components of the state vector for the time t_1, calculated along the initial orbit.

If we can not use the linearity then the basic tool for generating the transfer trajectories is Lambert's problem, which can find the transfer orbit between two positions for the given time interval of the transfer.

The interval of the possible application of the first or the second burn can be limited for the following cases:

– the initial orbit is an eccentric one and the target orbit is near-circular, the inclinations of the orbits coincide;

– the initial orbit is a near-circular one and the target orbit is eccentric, the inclinations coincide;

– the inclinations of the initial and target orbits are different and the semi-major axes are virtually the same.

Let us consider the major idea of the algorithm based on linearity. Let for the time t_1 we know the state vector of initial orbit $\left(\mathbf{r}_S(t_1), \mathbf{v}_S(t_1)\right)^T$ and the matrix of partial derivatives of the components of initial orbit for the time t_2 with respect to the components of the state vector for the time t_1:

$$\Phi(t_2, t_1) = \frac{\partial\left(\mathbf{r}_S(t_2), \mathbf{v}_S(t_2)\right)}{\partial\left(\mathbf{r}_S(t_1), \mathbf{v}_S(t_1)\right)} \tag{52}$$

If at the time t_1 we apply the impulse $\Delta\mathbf{V}_1$, and at the time t_2 - the impulse $\Delta\mathbf{V}_2$, the state vector for the time t_2 can be approximately represented in the shape:

$$\begin{pmatrix} \mathbf{r}_S(t_2) \\ \mathbf{v}_S(t_2) \end{pmatrix} + \Phi(t_2, t_1)\begin{pmatrix} 0 \\ \Delta\mathbf{V}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\mathbf{V}_2 \end{pmatrix} \tag{53}$$

If at the time t_2 the spacecraft transferred to the target orbit, the approximate equality is valid:

$$\begin{pmatrix} \mathbf{r}_T(t_2) \\ \mathbf{v}_T(t_2) \end{pmatrix} \approx \begin{pmatrix} \mathbf{r}_S(t_2) \\ \mathbf{v}_S(t_2) \end{pmatrix} + \Phi(t_2, t_1)\begin{pmatrix} 0 \\ \Delta\mathbf{V}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\mathbf{V}_2 \end{pmatrix} \tag{54}$$

We will consider this equality as an equation for the determination of the vectors $\Delta\mathbf{V}_1$ and $\Delta\mathbf{V}_2$. Thus, assuming that the thrusts have been performed by the times t_1 and t_2, we can evaluate them and determine the change in characteristic velocity: $\Delta V_\Sigma = \left|\Delta\mathbf{V}_1\right| + \left|\Delta\mathbf{V}_2\right|$. The times of the thrusts are determined by scanning the interval of their possible application under the condition of the minimum consumption of characteristic velocity.

Algorithm for evaluation of two burn maneuvers using linearity

Input information:

$t_{E1}, \left(\mathbf{r}_{EI}, \mathbf{v}_{EI}\right)^T$	–	the time and the state vector of initial orbit for this time;
$\left(t_{1L}, t_{2L}\right)$	–	interval of possible thrusts for transition from initial orbit to the transfer one;
t_{E2}, $\left(\mathbf{r}_{E2}, \mathbf{v}_{E2}\right)^T$	–	the time and the state vector of the target orbit for this time;
$\left(t_{1R}, t_{2R}\right)$	–	interval of possible thrusts for transition from transfer orbit to the target one;

Output information:

t_{I1}	–	evaluation of the time of the first thrust;
$\Delta\mathbf{V}_{I1}$	–	evaluation of the first thrust;
t_{I2}	–	evaluation of the time of the second thrust;
$\Delta\mathbf{V}_{I2}$	–	evaluation of the second thrust.

Designations

$\left(\mathbf{r}_S\left(t\right), \mathbf{v}_S\left(t\right)\right)$	–	state vector of the initial orbit for the time t;
$\left(\mathbf{r}_T\left(t\right), \mathbf{v}_T\left(t\right)\right)$	–	state vector of the target orbit for the time t;

Algorithm

1. For the time interval $\left(t_{1R}, t_{2R}\right)$ using the orbital parameters of the target orbit we calculate the table of the state vectors with fixed step: $\left\{t_{T,i}, \mathbf{r}_T\left(t_{T,i}\right), \mathbf{v}_T\left(t_{T,i}\right)\right\}$. We will consider that this table comprises N_T elements.

2. For each time $t_{T,i}$, $i = 1, ..., N_T$ using the orbital parameters of the initial orbit we calculate the matrix of partial derivatives of the components of the state vector for the time $t_{T,i}$ with respect to the components of the state vector for the time t_{E1}:

$$\Phi\left(t_{T,i}, t_{E1}\right) = \frac{\partial\left(\mathbf{r}_S\left(t_{T,i}\right), \mathbf{v}_S\left(t_{T,i}\right)\right)}{\partial\left(\mathbf{r}_S\left(t_{E1}\right), \mathbf{v}_S\left(t_{E1}\right)\right)} \tag{55}$$

and the vector of residuals between the state vectors in the target and initial orbits for the time $t_{T,i}$:

$$\begin{pmatrix} \Delta\mathbf{r}\left(t_{T,i}\right) \\ \Delta\mathbf{v}\left(t_{T,i}\right) \end{pmatrix} = \begin{pmatrix} \mathbf{r}_T\left(t_{T,i}\right) \\ \mathbf{v}_T\left(t_{T,i}\right) \end{pmatrix} - \begin{pmatrix} \mathbf{r}_S\left(t_{T,i}\right) \\ \mathbf{v}_S\left(t_{T,i}\right) \end{pmatrix} \tag{56}$$

3. Within the interval $\left(t_{1L},t_{2L}\right)$ using parameters of the initial orbit we calculate with permanent step the state vectors and the matrix of partial derivatives of the state vector for the initial time with respect to the state vector for the current time. The results are saved in the table. We will consider that the table comprises N_L elements. In this table for each time $t_{L,j}$, $j=1,...,N_L$ we have:

$$\mathbf{r}_S\left(t_{L,j}\right),\mathbf{v}_S\left(t_{L,j}\right) \text{ and } \Phi\left(t_{E1},t_{L,j}\right)=\frac{\partial\left(\mathbf{r}_S\left(t_{E1}\right),\mathbf{v}_S\left(t_{E1}\right)\right)}{\partial\left(\mathbf{r}_S\left(t_{L,j}\right),\mathbf{v}_S\left(t_{L,j}\right)\right)} \tag{57}$$

4. Then we perform the enumerative calculations for $i=1,...,N_T$ and $j=1,...,N_L$. For each step of the enumeration we calculate the matrix:

$$\Phi\left(t_{T,i},\, t_{L,j}\right)=\Phi\left(t_{T,i},t_{E1}\right)\Phi\left(t_{E1},t_{L,j}\right) \tag{58}$$

and solve the equation:

$$\begin{pmatrix} \Delta\mathbf{r}\left(t_{T,i}\right) \\ \Delta\mathbf{v}\left(t_{T,i}\right) \end{pmatrix}=\Phi\left(t_{T,i},\, t_{L,j}\right)\begin{pmatrix} 0 \\ \Delta\mathbf{V}_1 \end{pmatrix}+\begin{pmatrix} 0 \\ \Delta\mathbf{V}_2 \end{pmatrix} \tag{59}$$

If the matrix $\Phi\left(t_{T,i},\, t_{L,j}\right)$ is represented in the shape of block matrix:

$$\Phi\left(t_{T,i},\, t_{L,j}\right)=\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \tag{60}$$

the considered equation splits into two equations:

$$\Delta\mathbf{r}\left(t_{T,i}\right)=\varphi_{11}\Delta\mathbf{V}_1 \;,\; \Delta\mathbf{v}\left(t_{T,i}\right)=\varphi_{22}\;\Delta\mathbf{V}_1+\Delta\mathbf{V}_2 \tag{61}$$

We can find the solution of this system of two equations using formulas:

$$\Delta\mathbf{V}_1\left(t_{L,j}\right)=\varphi_{11}{}^{-1}\Delta\mathbf{r}\left(t_{T,i}\right),\quad \Delta\mathbf{V}_2\left(t_{T,i}\right)=\Delta\mathbf{v}\left(t_{T,i}\right)-\varphi_{22}\;\Delta\mathbf{V}_1 \tag{62}$$

The result of the enumerative search in the pair of indices i_m, j_m, for which the sum of the modules of impulses: $\left|\Delta\mathbf{V}_1\right|+\left|\Delta\mathbf{V}_2\right|$ reaches the minimum.

5. Then the result is generated:

$$\begin{aligned} t_{I\,1}&=t_{L,j_m}, \qquad \Delta\mathbf{V}_{I1}=\Delta\mathbf{V}_1\left(t_{L,j}\right), \\ t_{I\,21}&=t_{T,i_m}, \qquad \Delta\mathbf{V}_{I2}=\Delta\mathbf{V}_2\left(t_{T,i}\right). \end{aligned} \tag{63}$$

Algorithm for evaluation of the two burn maneuvers in the case of significant change of orbital parameters

In the cases when the maneuvering changes the orbital parameters to the extent that we can not use linearity, the thrusts have such values that making their evaluation we can neglect the non-central features of the Earth gravitational field. In this case we can use the Lambert’s problem. If a precise estimate is required the solution obtained for Lambert’s problem is the initial approximation for the iterative scheme. The algorithm for evaluation of two burn maneuvers using Lambert's problem is similar to the procedure considered above. The difference is that instead of solving the Eq. (54) we solve Lambert's problem.

Using the evaluation of the time of the transfer from the initial to the target orbit for reducing the enumerative search

1. Minimum time for the transfer is calculated by the formula:

$$t_{trn} = \pi\sqrt{\frac{a_{trn}^3}{\mu}} \tag{64}$$

where

$a_{trn} = \frac{a_1 + a_2}{2}$	-	semi-major axis of the transfer orbit,
a_1	-	semi-major axis of orbit 1,
a_2	-	semi-major axis of orbit 2.

2. Correction of the interval for the search of maneuvers. Denote the initial interval for the search as $\left[t_1, t_2\right]$. If $t_2 - t_1 < 2t_{trn}$, then the correction of the boundaries of the interval is needed. Correction of the boundaries of the search interval is performed using the following procedure. We find the time $t_{\min}$, corresponding to the minimum distance between the orbits. It is expedient to use for $t_{\min}$ the value determined by evaluation of the one burn maneuvers. If $t_1 > t_{\min} - t_{trn}$, we change t_1 ; and set it to $t_{\min} - t_{trn}$. If $t_2 < t_{\min} + t_{trn}$, we have to change t_2. t_2 is set to $t_{\min} + t_{trn}$.
3. For reduction of the enumerative search we should analyze only such times for the 1st and the 2nd thrusts for which the time interval between them exceeds t_{trn}.

Using the features of the initial and target orbits for reduction of the enumerative search

The intervals for the enumerative search can be reduced in special cases. These are the following cases:

– transfer from the eccentric orbit to the near-circular one without change of the inclination (circling the orbit);
– transfer from the near-circular orbit to the eccentric one without change of the inclination;
– change of the inclination without significant change of the semi-major axis (less than 1000 km).

Transfer from the eccentric orbit to the near-circular one without change of the inclination

The 1st thrust is looked for in the vicinity:

$$180^0 - \Delta\upsilon \le \upsilon_1 \le 180^0 + \Delta\upsilon \tag{65}$$

where υ_1 is true anomaly of the 1st orbit, adjusted to the interval 0º - 360º.

Transfer from the near-circular orbit to the eccentric one

The 2nd thrust is looked for in the vicinity (of the apocenter):

$$180^0 - \Delta\upsilon \le \upsilon_2 \le 180^0 + \Delta\upsilon \qquad (66)$$

where υ_2 is true anomaly of the orbit 2, adjusted to the interval 0º - 360º.

Change of the inclination without significant change of the semi-major axis (less than 1000 km).

The 1st thrust is looked for within the following intervals:

$$\begin{aligned} &0 \le u_1 \le \Delta u, \\ &180^0 - \Delta u \le u_1 \le 180^0 + \Delta u, \\ &360^0 - \Delta u \le u_1 \le 360^0, \end{aligned} \qquad (67)$$

where u_1 is the argument of latitude of the spacecraft in the orbit 1.

Parameter $\Delta\upsilon$ determines the vicinity of the apocenter and parameter Δu — the vicinity of the node. The default values of these parameters: 90º. Using the interactive mode these values can be changed.

The algorithm for evaluation of two burn maneuvers for the case of simultaneous change of semi-major axis and inclination.

Input information:

t_{E1}, $(\mathbf{r}_{E1}, \mathbf{v}_{E1})^T$ – the time and the state vector of the initial orbit for this time;

t_{E2}, $(\mathbf{r}_{E2}, \mathbf{v}_{E2})^T$ – the time and the state vector of the target orbit for this time.

Output information:

$t_{a1}, \Delta\mathbf{V}_{a1}$ – the time and the components of the 1st thrust,

$t_{a2}, \Delta\mathbf{V}_{a2}$ – the time and the components of the 2nd thrust.

Algorithm

1. Determination of the node closest to the pericenter of the initial orbit. Calculation of the time of passing this node. Operations described by items 1.1, 1.2 and 1.3 are performed for this purpose.

1.1. Calculation of the elements of the initial orbit: $a, e, \omega, i, \Omega,$ argument of latitude u using state vector $(\mathbf{r}_{E1}, \mathbf{v}_{E1})^T$ for the time t_{E1}.

1.2. Calculation of true anomalies υ_0 and υ_π for arguments of latitude 0 and π:

$$\upsilon_0 = -\omega, \quad \upsilon_\pi = \pi - \omega \qquad (68)$$

Adjustment of υ_0 and υ_π to the interval: $[-\pi, \pi]$. Calculation of the argument of latitude for the node which is closer to the pericenter using the formula:

$$u_{1i} = \begin{cases} 0, & \text{if } |\upsilon_0| < |\upsilon_\pi|, \\ \pi, & \text{otherwise.} \end{cases} \tag{69}$$

1.3. Calculation of the time for which the argument of latitude is u_{1i} . For this purpose we calculate Δt_{1i} the time of the transfer from the point with argument of latitude u_{1i} to the point with argument of latitude u. The time for which the argument of latitude is equal to u_{1i} is calculated by the formula: $t_{1i} = t_1 - \Delta t_{1i}$.

2. Calculation of the period of the initial orbit by the formula:

$$T_1 = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{70}$$

3. Preliminary calculation of the time interval of the transfer. Operations 3.1 – 3.4 are performed for this purpose.

3.1. Calculation of the state vector of the initial orbit for the time t_{1i} : $\left(\mathbf{r}(t_{1i}), \mathbf{v}(t_{1i})\right)^T$.

3.2. Calculation of the orbital elements: $a_2, e_2, \omega_2, i_2, \Omega_2$ using the state vector $\left(\mathbf{r}_{E2}, \mathbf{v}_{E2}\right)^T$ for the time t_{E2}.

3.3. Calculation of the semi-major axis of the transfer orbit:

$$a_{trn} = \frac{a_2 + |\mathbf{r}(t_{1i})|}{2} \tag{71}$$

3.4. Calculation of the time of transfer:

$$t_{trn} = \pi\sqrt{\frac{a_{trn}^3}{\mu}} \tag{72}$$

4. Evaluation of the times of the 1st and the 2nd thrusts. Operations described in 4.1-4.5.

4.1. Enumeration of the possible times of the 1st thrust:

$$t_{1k} = t_{1i} + kT_1, k = -2, -1, 0, 1, 2, 3 \tag{73}$$

4.2. determination of the time of the 2nd thrust:

$$t_{2k} = t_{1k} + T_{trn} \tag{74}$$

4.3. Calculation of the state vector $(\mathbf{r}_{1k}, \mathbf{v}_{1k})$ for the time t_{1k} using initial orbit. Calculation of the state vector $(\mathbf{r}_{2k}, \mathbf{v}_{2k})$ for the time t_{2k} using the target orbit.

Determination of the elements of the orbit for the transfer from position $\mathbf{r}_{1k}$ for the time t_{1k} to the position $\mathbf{r}_{2k}$ for the time t_{2k} using Lambert's problem procedure.

Calculation of the state vector $(\mathbf{r}_{1tk}, \mathbf{v}_{1tk})$ for the time t_{1k} and the state vector $(\mathbf{r}_{2tk}, \mathbf{v}_{2tk})$ for the time t_{2k} using the elements of the transfer orbit .

Calculation of the module of the resulting impulse:

$$m_k = |\mathbf{v}_{1k} - \mathbf{v}_{1tk}| + |\mathbf{v}_{2k} - \mathbf{v}_{2tk}| \quad (75)$$

4.4. Selection of the k, for which we have the minimum m_k. The times corresponding to m_k will be denoted as t_{1m} and t_{2m}.

5. Updating of the times of the 1st and the 2nd thrusts

5.1. Enumerative search for t_{1a} within $\left[t_{1m} - \frac{t_{trn}}{4}, t_{1m} + \frac{t_{trn}}{4}\right]$ with the step 5 minutes.

5.2. Enumerative search for t_{2a} within $\left[t_{2m} - \frac{t_{trn}}{4}, t_{2m} + \frac{t_{trn}}{4}\right]$ with the step 5 minutes.

5.3. For each pair of the values of t_{1a} and t_{2a} we perform the operations similar to those of item 4.3. We calculate the state vector for the initial orbit for the time t_{1a} and the target one – for the time t_{2a}. Then solve the Lambert's problem and determine the transfer impulses. Then calculate the module of resulting impulse.

5.4. Select the pair of times t_{1a} and t_{2a}, for which the minimum of resulting impulse is attained and generate the result.

ALGORITHM FOR EVALUATION OF THE ACCELERATION AND THE TIMES OF THE SWITCH-ON AND CUTOFF OF THE LOW THRUST ENGINE BY TWO INITIAL CONDITIONS

This section describes the algorithm for determining the time interval of the operation of the low thrust engine and the generated acceleration providing the transfer between the given orbits.

Input data:

t_{E1}, $(\mathbf{r}_{E1}, \mathbf{v}_{E1})^T$	–	time and the state vector of the initial orbit for this time;
(t_{1L}, t_{2L})	–	interval of possible engine switch-on;
t_{E2}, $(\mathbf{r}_{E2}, \mathbf{v}_{E2})^T$	–	time and the state vector of the target orbit for this time;
(t_{1R}, t_{2R})	–	interval of possible engine cutoff; we assume that the end of the interval of possible switch-on is does not exceed the beginning of the cutoff interval;
$h_{\max}$	–	maximum possible value of the step.

Output information:

t_{I1}	–	estimate of the switch-on time;
t_{I2}	–	estimate of the cutoff time;
Frame	–	sign of coordinate frame: 0 – inertial, 1 – orbital;.
$\mathbf{a} = (a_x, a_y, a_z)^T$	–	acceleration vector.

Designation

$\mathbf{y}_S(t) = (\mathbf{r}_S(t), \mathbf{v}_S(t))^T$ – state vector of initial orbit for the time t;

$\mathbf{x}_T(t) = (\mathbf{r}_T(t), \mathbf{v}_T(t))^T$ – state vector of target orbit for the time t;

Basic relationships

The time within the interval of possible switch-on (t_{1L}, t_{2L}), will be denoted as $t_{L,j}$, and the time of cutoff from the interval (t_{1R}, t_{2R}) — interval of possible cutoff - as $t_{R,i}$.

For the set times of switch-on and cutoff of low thrust engines: $t_{L,j}, t_{R,i}$ we determine the acceleration vector under the condition of minimum weighted residual between the state vectors: $\mathbf{x}_i = \mathbf{x}_T(t_{R,i})$ and

$$\mathbf{y}_i + \int_{t_{L,j}}^{t_{R,i}} \Phi(t_{R,i}, \tau) \cdot \mathbf{B}(\tau) d\tau \cdot \mathbf{a} \qquad (76)$$

where

$\mathbf{x}_i$ – state vector of the spacecraft in the target orbit for the time $t_{R,i}$;

$\mathbf{y}_i$ – state vector of the spacecraft in the initial orbit for the time t_{Ri}, $\mathbf{y}_i = \mathbf{y}_T(t_{R,i})$;

$\mathbf{B}(\tau)$ – matrix 6x3, describing the direction of the thrust; if the direction of the thrust vector is permanent in the inertial coordinate frame, the matrix $\mathbf{B}(\tau) = \begin{pmatrix} 0 \\ \mathbf{E}_3 \end{pmatrix}$; if the direction of the thrust is permanent in the orbital coordinate frame, $\mathbf{B}(\tau) = \begin{pmatrix} 0 \\ [\mathbf{e}_r, \mathbf{e}_n, \mathbf{e}_b] \end{pmatrix}$;

$\mathbf{E}_3$ – unit matrix of the 3rd order 3;

$\mathbf{e}_r, \mathbf{e}_n, \mathbf{e}_b$ – unit vectors of the directions of the axes of RNB coordinate frame;

$\mathbf{a}$ – the sought acceleration vector;

$\dfrac{\partial \mathbf{y}(t)}{\partial \mathbf{y}(t_{E1})}$ – matrix of partial derivatives of the state vector for the time t with respect to the state vector for the time of initial conditions along the initial orbit.

$$\Phi(t_{R,i}, \tau) = \frac{\partial \mathbf{y}(t_{R,i})}{\partial \mathbf{y}(t_{E1})} \left(\frac{\partial \mathbf{y}(\tau)}{\partial \mathbf{y}(t_{E1})} \right)^{-1} \text{ – transition matrix}$$

The six-dimensional diagonal matrix $\mathbf{W}_g$ is used as the weighting matrix in the search of the minimum. Diagonal elements of this matrix are defined by the values of a priori RMS errors of the position and velocity components: σ_P and σ_V:

$$\mathbf{W}_g = \begin{pmatrix} \frac{1}{\sigma_P^2} E_3 & 0_3 \\ 0_3 & \frac{1}{\sigma_V^2} E_3 \end{pmatrix} \tag{77}$$

Here 0_3 - zero matrix of the 3rd order.

In case we use TLE the values of these RMS errors are: $\sigma_P = 100$ m/s, $\sigma_V = 1$ m/s.

If we denote

$$\mathbf{Q}_{j,i} = \int_{t_{L,j}}^{t_{R,i}} \Phi\left(t_{R,i}, \tau\right) \mathbf{B}\left(\tau\right) d\tau \tag{78}$$

the considered weighted residual is represented in shape of the following quadratic form:

$$F = \left(\mathbf{x}_i - \mathbf{y}_i - \mathbf{Q}_{j,i}\mathbf{a}\right)^T \mathbf{W}_g \left(\mathbf{x}_i - \mathbf{y}_i - \mathbf{Q}_{j,i}\mathbf{a}\right) \tag{79}$$

For the fixed indexes i, j the vector $\mathbf{a}$, for which the quadratic form (79) reaches its minimum, can be found using the formula:

$$\mathbf{a}_{\min} = \left(\mathbf{Q}_{j,i}^T \mathbf{W}_g \mathbf{Q}_{j,i}\right)^{-1} \mathbf{Q}_{j,i}^T \mathbf{W}_g \left(\mathbf{x}_i - \mathbf{y}_i\right) \tag{80}$$

The respective minimum value of the quadratic form is equal to:

$$F_{\min} = \left(\mathbf{x}_i - \mathbf{y}_i - \mathbf{Q}_{j,i}\mathbf{a}_{\min}\right)^T \mathbf{W}_g \left(\mathbf{x}_i - \mathbf{y}_i - \mathbf{Q}_{j,i}\mathbf{a}_{\min}\right) \tag{81}$$

The global minimum of the quadratic form (79) can be found by scanning in i and j. For reduction of computation for the enumerative search we need the recursive formulas connecting matrices $\mathbf{Q}_{j,i+1}$ and $\mathbf{Q}_{j-1,i}$ with the matrix $\mathbf{Q}_{i,j}$. Let us derive these formulas. Actually,

$$\begin{gathered} \mathbf{Q}_{j,i+1} = \int_{t_{L,j}}^{t_{R,i+1}} \Phi\left(t_{R,i+1}, \tau\right) \mathbf{B}\left(\tau\right) d\tau = \Phi\left(t_{R,i+1}, t_{R,i}\right) \int_{t_{L,j}}^{t_{R,i}} \Phi\left(t_{R,i}, \tau\right) \mathbf{B}\left(\tau\right) d\tau + \\ \int_{t_i}^{t_{R,i+1}} \Phi\left(t_{i+1}, \tau\right) \mathbf{B}\left(\tau\right) d\tau = \Phi\left(t_{R,i+1}, t_{R,i}\right) \mathbf{Q}_{j,i} + \int_{t_{R,i}}^{t_{R,i+1}} \Phi\left(t_{R,i+1}, \tau\right) \mathbf{B}\left(\tau\right) d\tau \end{gathered} \tag{82}$$

$$\mathbf{Q}_{j-1,i} = \int_{t_{L,j-1}}^{t_{R,i}} \Phi\left(t_{R,i},\tau\right)\mathbf{B}\left(\tau\right)d\tau = \int_{t_{L,j-1}}^{t_{L,j}} \Phi\left(t_{R,i},\tau\right)\mathbf{B}\left(\tau\right)d\tau + \int_{t_{L,j}}^{t_{R,i}} \Phi\left(t_{R,i},\tau\right)\mathbf{B}\left(\tau\right)d\tau =$$

$$\Phi\left(t_{R,i},t_{L,j}\right) \int_{t_{L,j-1}}^{t_{L,j}} \Phi\left(t_{L,j},\tau\right)\mathbf{B}\left(\tau\right)d\tau + \mathbf{Q}_{j,i} \tag{83}$$

Using the trapezoid formulas for approximate calculation of the integrals $\int_{t_{R,i}}^{t_{R,i+1}} \Phi\left(t_{R,i+1},\tau\right)\mathbf{B}\left(\tau\right)d\tau$ and $\int_{t_{L,j-1}}^{t_{L,j}} \Phi\left(t_{L,j},\tau\right)\mathbf{B}\left(\tau\right)d\tau$ yields:

$$\mathbf{Q}_{j,i+1} \approx \Phi\left(t_{R,i+1},t_{R,i}\right)\mathbf{Q}_{j,i} + \frac{t_{R,i+1}-t_{R,i}}{2}\left[\mathbf{B}\left(t_{R,i+1}\right) + \Phi\left(t_{R,i+1},t_{R,i}\right)\mathbf{B}\left(t_{R,i}\right)\right] \tag{84}$$

$$\mathbf{Q}_{j-1,i} \approx \frac{1}{2}\Phi\left(t_{R\,i},t_{L\,j}\right)\left(B\left(t_{L\,j}\right) + \Phi\left(t_{L\,j},t_{L\,j-1}\right)B\left(t_{L\,j}\right)\right)\cdot\left(t_{L\,j} - t_{L\,j-1}\right) + Q_{j,i} \tag{85}$$

Sequence of the operations

1. Select the number of fragmentations N of the interval $\left(t_{1R},t_{2R}\right)$ of possible engine cutoff under the condition: $\frac{t_{2R}-t_{1R}}{N-1} < h_{\max}$ and calculate the value of the step for the interval of possible engine cutoff:

$$h_R = \frac{t_{2R}-t_{1R}}{N-1} \tag{86}$$

2. Within the interval $\left(t_{1R},t_{2R}\right)$ using parameters of the target orbit with permanent step h_R calculate the table of state vectors:

$$\left\{\mathbf{x}_i = \mathbf{x}_T\left(t_{R,i}\right),\ t_{R,i} = t_{1R} + (i-1)h_R\right\},\ \ i = 1,...N \tag{87}$$

3. For each time $t_{R,i}$, $i = 1,...,N$ using parameters of the initial orbit we calculate the matrix of the derivatives of the components of the state vector for the time $t_{R,i}$ with respect to the components of the state vector for the initial time t_{E1}:

$$\mathbf{Z}\left(t_{R,i}\right) = \frac{\partial\mathbf{y}\left(t_{R,i}\right)}{\partial\mathbf{y}\left(t_{1E}\right)} \tag{88}$$

4. Then select the number of fragmentations M of the interval of possible switch-on of the engine $\left(t_{1L},t_{2L}\right)$ under the condition that the step of the fragmentation $h_L = \frac{t_{2L}-t_{1L}}{M-1}$ does not exceed $h_{\max}$.

5. Within the interval of possible engine switch-on (t_{1L}, t_{2L}) using the parameters of the initial orbit with permanent step h_L we calculate the table comprising the time $t_{L,j} = t_{1L} + (j-1)h_L,\ j = 1,\ldots,M$, state vector of the initial orbit for this time and the matrix of partial derivatives of the components of the state vector for the initial time t_{E1} with respect to the components of the state vector for the time $t_{L,j}$

$$\left\{ t_{L,j},\ \mathbf{y}_j,\ \mathbf{Z}_j^{-1} = \left(\frac{\partial \mathbf{y}\left(t_{L,j}\right)}{\partial \mathbf{y}\left(t_{\mathbf{E}}\right)} \right)^{-1} = \frac{\partial \mathbf{y}\left(t_{E1}\right)}{\partial \mathbf{y}\left(t_{L,j}\right)} \right\}$$

Further operation described by items 6-12 are performed for two variants of the direction of the thrust vector: permanent in the inertial coordinate frame and permanent in the orbital coordinate frame.

6. Calculate the matrix $\mathbf{Q}_{M,1}$ by performing the following operations. If $t_{M,L} = t_{1R}$, then matrix $\mathbf{Q}_{M,1} = 0$. If $t_{M,L} < t_{1R}$, the interval $\left[t_{M,L}, t_{1R}\right]$ is fragmented into several parts in a way that the length of no part exceeds $h_{\max}$ (see the procedure for selecting the step in items 1 or 3). The matrix $\mathbf{Q}_{M,1} = \int_{t_{ML}}^{t_{1R}} \Phi\left(t_{1R}, \tau\right)\mathbf{B}\left(\tau\right)d\tau$ is calculated using trapezium formula for each part.

7. Determine the values of the following variables:

F_c – current minimum value of the quadratic form,

$\mathbf{a}_c$ – acceleration vector corresponding to the current minimum value of the quadratic form,

$i_c,\ j_c$ – indices, corresponding to the current minimum value of the quadratic form.

Initial value $F_c = +\infty$

8. We organize two cycles. The external cycle for j from M until 1. The internal cycle for i from 1 until N.

9. Prior to begin of the internal cycle we set the variable matrix $\mathbf{Q}_c = \mathbf{Q}_{j1}$.

10. Internal cycle. Using the matrix $\mathbf{Q}_c$, by Eqs (80) and (81) we find the acceleration vector $\mathbf{a}$ and the value of functional F. If $F < F_c$, new values are assigned to the variables $F_c, \mathbf{a}_c, i_c, j_c$. Using Eq. (84) and the current matrix $\mathbf{Q}_c = \mathbf{Q}_{j,i}$ we calculate the new matrix $\mathbf{Q}_c$:

$$\mathbf{Q}_c = \mathbf{Q}_{j,i+1} = \Phi\left(t_{R,i+1}, t_{Ri}\right)\mathbf{Q}_{j,i} + \frac{1}{2}\left(\mathbf{B}\left(t_{R,i+1}\right) + \Phi\left(t_{R,i+1}, t_{R,i}\right)\mathbf{B}\left(t_{R,i}\right)\right)\cdot\left(t_{R,i+1} - t_{R,i}\right) \qquad (89)$$

11. After the completion of the internal cycle, using the recurrent formula (4.10) we calculate the new matrix $\mathbf{Q}_c$:

$$\mathbf{Q}_c = \mathbf{Q}_{j-1,1} = \frac{1}{2}\Phi\left(t_{R1}, t_{L,j}\right)\left(\mathbf{B}\left(t_{L,j}\right) + \Phi\left(t_{L,j}, t_{L,j-1}\right)\mathbf{B}\left(t_{L,j}\right)\right)\cdot\left(t_{L,j} - t_{L,j-1}\right) + \mathbf{Q}_{j,1} \qquad (90)$$

12. We compare the values of F_c, calculated for the two variants of the direction of the thrust vector. Then select the minimum value and generate the result using $\mathbf{a}_c$, i_c, j_c.

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